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# Phase retrieval from the norms of affine transformations ☆

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ABSTRACT

In this paper, we consider the generalized phase retrieval from affine measurements. This problem aims to recover signals $\boldsymbol{x} \in \mathbb{F}^d$ from the magnitude of the affine transformations $y_j = \|M_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2^2,\ j = 1, \ldots, m$, where $M_j \in \mathbb{F}^{d\times r}, \boldsymbol{b}_j \in \mathbb{F}^r, \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and we call it *generalized affine phase retrieval*. We first develop a framework for generalized affine phase retrieval with presenting several necessary and sufficient conditions for $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ having generalized affine phase retrieval property. Next, we focus on the minimal measurement number problem and establish some results for it. Particularly, we show if $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{F}^{d\times r} \times \mathbb{F}^r$ has generalized affine phase retrieval property, then $m \geq d + \lfloor d/r \rfloor$ for $\mathbb{F} = \mathbb{R}$ ($m \geq 2d + \lfloor d/r \rfloor$ for $\mathbb{F} = \mathbb{C}$). We also show that the lower bounds are tight provided $r \mid d$. These results imply that one can reduce the measurement number by raising $r$, i.e. the rank of $M_j$. This highlights a notable difference between generalized affine phase retrieval and generalized phase retrieval. Furthermore, using tools of algebraic geometry, we show that $m \geq 2d$ (resp.

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$m \geq 4d-1$) generic measurements $\mathcal{A} = \{(M_j, b_j)\}_{j=1}^m$ have the generalized phase retrieval property for $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$).


## 1. Introduction

### 1.1. Phase retrieval

Phase retrieval aims to recover a signal $\boldsymbol{x} \in \mathbb{F}^d$ from the measurements $|\langle \mathbf{a}_j, \boldsymbol{x} \rangle|$, $j = 1, \ldots, m$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbf{a}_j \in \mathbb{F}^d$ are the measurement vectors. It arises in many areas such as X-ray crystallography [13,16], microscopy [15], astronomy [8], coherent diffractive imaging [17,11] and optics [19]. To state conveniently, set $A := (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ and $\mathbf{M}_A(\boldsymbol{x}) := (|\langle \mathbf{a}_1, \boldsymbol{x} \rangle|, \ldots, |\langle \mathbf{a}_m, \boldsymbol{x} \rangle|) \in \mathbb{R}^m$. Noting that for any $c \in \mathbb{F}$ with $|c| = 1$ we have $\mathbf{M}_A(\boldsymbol{x}) = \mathbf{M}_A(c\boldsymbol{x})$ and hence we can only hope to recover $\boldsymbol{x}$ up to a unimodular constant. If $\mathbf{M}_A(\boldsymbol{x}) = \mathbf{M}_A(\boldsymbol{y})$ implies $\boldsymbol{x} \in \{c\boldsymbol{y} : c \in \mathbb{F}, |c| = 1\}$, then we say $A$ has *phase retrieval property* in $\mathbb{F}^d$. A fundamental problem in phase retrieval is to give the minimal $m$ for which there exists $A = (\mathbf{a}_1, \ldots, \mathbf{a}_m)^\top \in \mathbb{F}^{m \times d}$ such that it has phase retrieval property in $\mathbb{F}^d$. For the case $\mathbb{F} = \mathbb{R}$, it is well known that the minimal measurement number $m$ is $2d - 1$ [1]. For the complex case $\mathbb{F} = \mathbb{C}$, this question remains open. Specifically, Conca, Edidin, Hering and Vinzant [6] prove that $m \geq 4d - 4$ generic measurement vectors $A = (\mathbf{a}_1, \ldots, \mathbf{a}_m)^\top \in \mathbb{C}^{m \times d}$ have phase retrieval property in $\mathbb{C}^d$ and they furthermore show $4d - 4$ is sharp if $d$ is in the form of $2^k + 1$, $k \in \mathbb{Z}_+$. In [18], for the case where $\mathbb{F} = \mathbb{C}$ and $d = 4$, Vinzant present $11 = 4d - 5 < 4d - 4$ measurement vectors and prove they have phase retrieval property in $\mathbb{C}^4$, which implies $4d - 4$ is not sharp for some dimension $d$. Beyond the minimal measurement number problem, many efficient algorithms have also been developed for recovering $\boldsymbol{x}$ from $\mathbf{M}_A(\boldsymbol{x})$ (see [4,5,10]).

### 1.2. Generalized phase retrieval and affine phase retrieval

A generalized version of phase retrieval, termed *generalized phase retrieval*, was introduced by Wang and Xu [20]. For generalized phase retrieval, one aims to reconstruct $\boldsymbol{x} \in \mathbb{F}^d$ through quadratic samples $\boldsymbol{x}^* A_1 \boldsymbol{x}, \ldots, \boldsymbol{x}^* A_m \boldsymbol{x}$ where $A_j \in \mathbb{F}^{d \times d}$ are Hermitian matrix for $\mathbb{F} = \mathbb{C}$ (symmetric matrix for $\mathbb{F} = \mathbb{R}$). Set $\mathcal{A} := (A_j)_{j=1}^m$ and $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) := (\boldsymbol{x}^* A_1 \boldsymbol{x}, \ldots, \boldsymbol{x}^* A_m \boldsymbol{x})$. We say $\mathcal{A}$ has *generalized phase retrieval property* if $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = \mathbf{M}_{\mathcal{A}}(\boldsymbol{y})$ implies that $\boldsymbol{x} \in \{c\boldsymbol{y} : c \in \mathbb{F}, |c| = 1\}$. In [20], Wang and Xu show the fantastic connection among phase retrieval, nonsingular bilinear form and embedding. They also study the minimal $m$ for which there exists $\mathcal{A} = (A_j)_{j=1}^m$ such that it has generalized phase retrieval property. Particularly, they show that for the case where $\mathbb{F} = \mathbb{C}$, the measurement number $m$ obeys $m \geq 4d - 2 - 2\alpha$ where $\alpha$ denotes the number of 1's in the binary expansion of $d - 1$. If we take $A_j = \boldsymbol{a}_j \boldsymbol{a}_j^*$, then the generalized phase retrieval reduces to the standard phase retrieval. Similarly, if we require $A_j, j = 1, \ldots, m$, are orthogonal projection matrices, then the generalized phase retrieval reduces to phase retrieval by projection [3,2]. Hence, the generalized phase retrieval includes the standard phase retrieval and the phase retrieval by projection as a special case. Both standard phase retrieval and generalized phase retrieval require the measurement number is greater than or equal to $4d - 2 - 2\alpha$. However, as will be shown later, the minimal measurement number for generalized affine phase retrieval can be reduced to $2d+1$ by raising the rank of $A_j$, which highlights a notable difference between generalized affine phase retrieval and generalized phase retrieval.

Affine phase retrieval arised in holography [14] as well as in phase retrieval with background information [21], which aims to recover $\boldsymbol{x} \in \mathbb{F}^d$ from $|\langle \mathbf{a}_j, \boldsymbol{x}\rangle + b_j|,\ j = 1, \ldots, m$, where $\mathbf{a}_j \in \mathbb{F}^d$ and $b_j \in \mathbb{F}$. The authors of [9] develop the general framework of affine phase retrieval with highlighting the difference between affine phase retrieval and standard phase retrieval. Unlike the standard phase retrieval where one can only recover $\boldsymbol{x}$ up to a unimodular constant, it is possible to recover $\boldsymbol{x}$ exactly in affine phase retrieval. Particularly, for the case where $\mathbb{F} = \mathbb{C}$, the authors of [9] show that there exist $m = 3d$ measurements $\{(\mathbf{a}_j, b_j)\}_{j=1}^m$ so that one can recover $\boldsymbol{x}$ from $|\langle \mathbf{a}_j, \boldsymbol{x}\rangle + b_j|, j = 1, \ldots, m$. They furthermore prove the measurement number $3d$ is sharp for recovering $\boldsymbol{x} \in \mathbb{C}^d$ from $|\langle \mathbf{a}_j, \boldsymbol{x}\rangle + b_j|, j = 1, \ldots, m$. Similarly, for the case where $\mathbb{F} = \mathbb{R}$, it was shown in [9] that $m = 2d$ measurements are sufficient and necessary for recovering $\boldsymbol{x}$ from $|\langle \mathbf{a}_j, \boldsymbol{x}\rangle + b_j|, j = 1, \ldots, m$.

### *1.3. Generalized affine phase retrieval*

In this paper, we consider the recovery of $\boldsymbol{x} \in \mathbb{F}^d$ from the affine quadratic measurements

$$y_j = \|M_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2^2, \quad j = 1, \ldots, m,$$

where $M_j \in \mathbb{F}^{d\times r}$ and $\boldsymbol{b}_j \in \mathbb{F}^r$. Set $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{F}^{d\times r} \times \mathbb{F}^r$, we can view $\mathcal{A}$ as a point in $\mathbb{F}^{m(d\times r)} \times \mathbb{F}^{mr}$. Define the map $\mathbf{M}_{\mathcal{A}} : \mathbb{F}^d \to \mathbb{R}^m$ by

$$\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = (\|M_1^* \boldsymbol{x} + \boldsymbol{b}_1\|_2^2, \ldots, \|M_m^* \boldsymbol{x} + \boldsymbol{b}_m\|_2^2). \tag{1.1}$$

Our goal is to study whether a signal $\boldsymbol{x} \in \mathbb{F}^d$ can be uniquely reconstructed from $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x})$. To state conveniently, we introduce the definition of the *generalized affine phase retrieval property*.

**Definition 1.1.** Let $r \in \mathbb{Z}_{\geq 1}$ and $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{F}^{d\times r} \times \mathbb{F}^r$. We say $\mathcal{A}$ has the generalized affine phase retrieval property if $\mathbf{M}_{\mathcal{A}}$ is injective on $\mathbb{F}^d$.

We next introduce the connection between generalized affine phase retrieval and generalized phase retrieval. It is easy to check that

$$y_j = \|M_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2^2 = \tilde{\boldsymbol{x}}^* A_j \tilde{\boldsymbol{x}},\ j = 1, \ldots, m, \tag{1.2}$$

where

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \qquad \text{and} \qquad A_j = \begin{pmatrix} M_j M_j^* & M_j \boldsymbol{b}_j \\ (M_j \boldsymbol{b}_j)^* & \boldsymbol{b}_j^* \boldsymbol{b}_j \end{pmatrix}.$$

Equation (1.2) shows that generalized affine phase retrieval can be reduced to recover $\tilde{\boldsymbol{x}} \in \mathbb{F}^{d+1}$ from $\tilde{\boldsymbol{x}}^* A_j \tilde{\boldsymbol{x}}, j = 1, \ldots, m$. Since the last entry of $\tilde{\boldsymbol{x}}$ is 1, we can recover $\tilde{\boldsymbol{x}}$ from $\tilde{\boldsymbol{x}}^* A_j \tilde{\boldsymbol{x}}, j = 1, \ldots, m$ exactly rather than up to a global phase. Hence, the generalized affine phase retrieval can be considered as the extension of both the generalized phase retrieval and the affine phase retrieval.

### *1.4. Continuous map*

Note that $\boldsymbol{x} \in \mathbb{R}^d$ has $d$ real variables ($2d$ real variables for the complex case). Naturally, one may be interested in whether it is possible to recover $\boldsymbol{x} \in \mathbb{R}^d$ from $d$ nonnegative measurements ($2d$ nonnegative measurements for $\mathbb{F} = \mathbb{C}$). We state the question as follows. For $j = 1, \ldots, m$, suppose that $f_j : \mathbb{F}^d \to \mathbb{R}_+$ is a continuous nonnegative function, i.e. $f_j(\boldsymbol{x}) \geq 0$. For $\boldsymbol{x} \in \mathbb{F}^d$, set

$$\mathbf{F}(\boldsymbol{x}) := (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})) \in \mathbb{R}_+^m. \tag{1.3}$$

One may be interested in the question: *What is the smallest $m$ so that $\mathbf{F}$ is injective on $\mathbb{R}^d$?* Under some mild conditions for $\mathbf{F}$, we show that $m \geq d+1$ is necessary for $\mathbf{F}$ being injective on $\mathbb{R}^d$ ($m \geq 2d+1$ on $\mathbb{C}^d$). As we will show later, there exists $\{(A_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{R}^{d\times d} \times \mathbb{R}^d$ with $m = d+1$ so that $\mathbf{M}_{\mathcal{A}}$ is injective on $\mathbb{R}^d$. This implies that the generalized affine phase retrieval can achieve the lower bound $m = d+1$. A similar conclusion also holds for the complex case.

### *1.5. Our contribution*

In this paper, we develop the framework of the generalized affine phase retrieval. Particularly, we focus on the number of measurements needed to achieve generalized affine phase retrieval. We first present some equivalent conditions and then study the minimal measurement number to guarantee the generalized affine phase retrieval property for both real and complex signals. For $\mathbb{F} = \mathbb{R}$, we show that $m \geq d + \lfloor \frac{d}{r} \rfloor$ ($m \geq 2d + \lfloor \frac{d}{r} \rfloor$ for $\mathbb{F} = \mathbb{C}$) is necessary for there existing measurements $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{F}^{d\times r} \times \mathbb{F}^r$ which has this property. We also show that the bounds are tight provided $d/r \in \mathbb{Z}$. Compared with the generalized phase retrieval, the generalized affine phase retrieval can reduce the measurement number heavily by raising the rank of $M_j$. This also highlights a notable difference between the generalized affine phase retrieval and generalized phase retrieval.

Using the tools developed in [1,6,20], we show that $m \geq 2d$ generic measurements $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{F}^{m(d\times r)\times mr}$ for $\mathbb{F} = \mathbb{R}$ ($m \geq 4d-1$ for $\mathbb{F} = \mathbb{C}$) can do generalized affine phase retrieval in $\mathbb{F}^d$.

## 2. The minimal measurement number for continuous map

Recall that $\mathbf{F} : \mathbb{F}^d \to \mathbb{R}_+^m$ is a continuous map. The next theorem shows that the necessary condition for $\mathbf{F}$ being injective is $m \geq d+1$ under some mild condition for $\mathbf{F}(\boldsymbol{x})$.

**Theorem 2.1.** *Suppose that $\mathbf{F} : \mathbb{F}^d \to \mathbb{R}_+^m$ is a continuous map which satisfies*

$$\lim_{R\to+\infty} \inf_{\|\boldsymbol{x}\|\geq R} \|\mathbf{F}(\boldsymbol{x})\| = +\infty. \tag{2.1}$$

*If $m = d$ for $\mathbb{F} = \mathbb{R}$ ($m = 2d$ for $\mathbb{F} = \mathbb{C}$), then $\mathbf{F}$ is not injective on $\mathbb{F}^d$.*

**Proof.** Note that $\mathbb{C}^d \cong \mathbb{R}^{2d}$. We just need to consider the case where $\mathbb{F} = \mathbb{R}$. To this end, we use $\mathbb{S}^d$ to denote the $d$-sphere and use $\mathcal{N}$ to denote the north pole of $\mathbb{S}^d$. Let $g : \mathbb{R}^d \to \mathbb{S}^d \setminus \{\mathcal{N}\}$ be the natural homeomorphism between $\mathbb{R}^d$ and $\mathbb{S}^d \setminus \{\mathcal{N}\}$. Then $\mathbf{F}_g := g \circ \mathbf{F} \circ g^{-1}$ is the operator which maps $\mathbb{S}^d \setminus \{\mathcal{N}\}$ to $g(\mathbb{R}_+^d) \subset \mathbb{S}^d \setminus \{\mathcal{N}\}$. Set

$$\widetilde{\mathbf{F}}_g(\boldsymbol{x}) := \begin{cases} \mathbf{F}_g(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{S}^d \setminus \{\mathcal{N}\} \\ \boldsymbol{x}, & \boldsymbol{x} = \mathcal{N} \end{cases}.$$

Since $\mathbf{F}$ satisfies (2.1), $\tilde{\mathbf{F}}_g$ is continuous on $\mathbb{S}^d$. Note that $\widetilde{\mathbf{F}}_g(\mathbb{S}^d) \subset g(\mathbb{R}_+^d) \cup \{\mathcal{N}\}$. Thus the range of $\widetilde{\mathbf{F}}_g$ is not the whole $\mathbb{S}^d$. Recall that if a single point is removed from a $d$-sphere, it becomes homeomorphic to $\mathbb{R}^d$, which means that $\widetilde{\mathbf{F}}_g(\mathbb{S}^d) \hookrightarrow \mathbb{R}^d$. We now get a continuous map from $\mathbb{S}^d$ to $\mathbb{R}^d$. We abuse the notation and still use $\widetilde{\mathbf{F}}_g$ to denote the map. By Borsuk-Ulam theorem, there exists $\{\boldsymbol{x}, -\boldsymbol{x}\} \subset \mathbb{S}^d$ such that $\widetilde{\mathbf{F}}_g(\boldsymbol{x}) = \widetilde{\mathbf{F}}_g(-\boldsymbol{x})$. Let $\boldsymbol{y}_1 = g^{-1}(\boldsymbol{x})$ and $\boldsymbol{y}_2 = g^{-1}(-\boldsymbol{x})$, and then $\mathbf{F}(\boldsymbol{y}_1) = \mathbf{F}(\boldsymbol{y}_2)$ since $g$ is injective. Now, we claim that $\boldsymbol{y}_1 \neq \infty$ and $\boldsymbol{y}_2 \neq \infty$. Indeed, if $\boldsymbol{y}_1 = \infty$, then $\boldsymbol{x} = \mathcal{N}$ since $\boldsymbol{x} = g(\boldsymbol{y}_1)$. Hence $-\boldsymbol{x}$ is the south pole which implies that $\mathbf{F}(\boldsymbol{y}_2)$ is finite since $\boldsymbol{y}_2 = g^{-1}(-\boldsymbol{x})$. Hence, we find two points $\boldsymbol{y}_1 \neq \boldsymbol{y}_2 \in \mathbb{R}^d$, but $\mathbf{F}(\boldsymbol{y}_1) = \mathbf{F}(\boldsymbol{y}_2)$, which arrives at the conclusion. □

**Remark 2.2.** In Theorem 2.1, we require that the image of $\mathbf{F} = (f_1, \ldots, f_m)$ is a subset of $\mathbb{R}_+^d$. If we remove the requirement of $f_j(\boldsymbol{x}) \geq 0$, then there exists a map $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ which is injective on $\mathbb{R}^d$. In fact, we just need to take $\mathbf{F}(\boldsymbol{x}) = (\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle, \ldots, \langle \boldsymbol{a}_d, \boldsymbol{x}\rangle)$ where $\boldsymbol{a}_j \in \mathbb{R}^d$ satisfying $\operatorname{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d\} = \mathbb{R}^d$, and then $\mathbf{F}$ is injective on $\mathbb{R}^d$. Moreover, if we remove the condition (2.1), we can set $\mathbf{F}(\boldsymbol{x}) := (\exp(x_1), \ldots, \exp(x_d))$ which is also injective on $\mathbb{R}^d$.

## 3. Generalized affine phase retrieval for real signals

In this section, we consider the generalized affine phase retrieval for real signals. We first state several equivalent conditions for the generalized affine phase retrieval. Suppose that $M \in \mathbb{R}^{d\times r}$ and $\boldsymbol{b} \in \mathbb{R}^r$. Then the following formula is straightforward to check:

$$\|M^\top \boldsymbol{x} + \boldsymbol{b}\|_2^2 - \|M^\top \boldsymbol{y} + \boldsymbol{b}\|_2^2 = 4\left(\boldsymbol{u}^\top MM^\top \boldsymbol{v} + (M\boldsymbol{b})^\top \boldsymbol{v}\right) \quad \text{for any } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d \tag{3.1}$$

where $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{x}+\boldsymbol{y})$ and $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{x}-\boldsymbol{y})$.

**Theorem 3.1.** *Suppose that $r \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{R}^{d\times r} \times \mathbb{R}^r$. Then the followings are equivalent:*

(1) *$\mathcal{A}$ has the generalize affine phase retrieval property in $\mathbb{R}^d$.*
(2) *For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ and $\boldsymbol{v} \neq 0$, there exists a $j$ with $1 \leq j \leq m$ such that*

$$\boldsymbol{u}^\top M_j M_j^\top \boldsymbol{v} + (M_j \boldsymbol{b}_j)^\top \boldsymbol{v} \neq 0.$$

(3) $\operatorname{span}\{M_j M_j^\top \boldsymbol{u} + M_j \boldsymbol{b}_j\}_{j=1}^m = \mathbb{R}^d$ *for any $\boldsymbol{u} \in \mathbb{R}^d$.*
(4) *The Jacobian of $\mathbf{M}_{\mathcal{A}}$ has rank $d$ for all $\boldsymbol{x} \in \mathbb{R}^d$.*

**Proof.** (1)⇔(2). Assume that there exist $\boldsymbol{x} \neq \boldsymbol{y}$ in $\mathbb{R}^d$ such that $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) - \mathbf{M}_{\mathcal{A}}(\boldsymbol{y}) = 0$. Then (3.1) means that for all $j$

$$\|M_j^\top \boldsymbol{x} + \boldsymbol{b}_j\|_2^2 - \|M_j^\top \boldsymbol{y} + \boldsymbol{b}_j\|_2^2 = 4(\boldsymbol{u}^\top M_j M_j^\top \boldsymbol{v} + (M_j \boldsymbol{b}_j)^\top \boldsymbol{v}) = 0.$$

Noting that $\boldsymbol{v} \neq 0$, we conclude a contradiction with (2). It means that (2) ⇒ (1). The converse also follows from the same argument.

(2)⇔(3). If for some $\boldsymbol{u}$ such that $\operatorname{span}\{M_j M_j^\top \boldsymbol{u} + M_j \boldsymbol{b}_j\}_{j=1}^m \neq \mathbb{R}^d$, then there exists a $\boldsymbol{v} \neq 0$ such that $\boldsymbol{v} \perp \operatorname{span}\{M_j M_j^\top \boldsymbol{u} + M_j \boldsymbol{b}_j\}_{j=1}^m$. It implies that $\boldsymbol{u}^\top M_j M_j^\top \boldsymbol{v} + (M_j \boldsymbol{b}_j)^\top \boldsymbol{v} = 0$ for all $j = 1, \ldots, m$. This is a contradiction. The converse clearly also holds.

(3)⇔(4). Note that the Jacobian $J(\boldsymbol{x})$ of the map $\mathbf{M}_{\mathcal{A}}$ at $\boldsymbol{x} \in \mathbb{R}^d$ is exactly

$$J(\boldsymbol{x}) = 2[M_1 M_1^\top \boldsymbol{x} + M_1 \boldsymbol{b}_1, \ldots, M_m M_m^\top \boldsymbol{x} + M_m \boldsymbol{b}_m].$$

Thus (3) is equivalent to that the rank of $J(\boldsymbol{x})$ is $d$ for all $\boldsymbol{x} \in \mathbb{R}^d$. □

**Corollary 3.2.** *Suppose that $r \in \mathbb{Z}_{\geq 1}$ and $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ where $(M_j, \boldsymbol{b}_j) \in \mathbb{R}^{d\times r} \times \mathbb{R}^r$. If $\mathcal{A}$ has generalized affine phase retrieval property in $\mathbb{R}^d$ then $m \geq d + \lfloor \frac{d}{r} \rfloor$.*

**Proof.** To this end, we just need to show that if $m \leq d + \lfloor \frac{d}{r} \rfloor - 1$, then $\mathcal{A}$ is not generalized affine phase retrievable in $\mathbb{R}^d$. When $r \geq d+1$, the conclusion follows from

(3) in Theorem 3.1 directly. Hence, we only consider the case where $r \le d$. We claim that there exists $\boldsymbol{u} \in \mathbb{R}^d$ such that $M_j(M_j^\top \boldsymbol{u} + \boldsymbol{b}_j) = 0$ for all $j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$. Thus, if $m \le \lfloor \frac{d}{r} \rfloor + d - 1$ then

$$\mathrm{span}\{M_j M_j^\top \boldsymbol{u} + M_j \boldsymbol{b}_j\}_{j=1}^m = \mathrm{span}\{M_j M_j^\top \boldsymbol{u} + M_j \boldsymbol{b}_j\}_{j=\lfloor \frac{d}{r} \rfloor + 1}^m \ne \mathbb{R}^d.$$

According to (3) in Theorem 3.1, we arrive at the conclusion.

It remains to prove the claim. For any $j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$, let $\boldsymbol{b}_j' \in \mathbb{R}^r$ be the orthogonal projection vector of $\boldsymbol{b}_j$ onto the space spanned by the rows of $M_j$. Then we have $M_j(\boldsymbol{b}_j - \boldsymbol{b}_j') = 0$ for all $j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$. On the other hand, since $\boldsymbol{b}_j'$ is in the space spanned by the columns of $M_j^\top$, it means that there exists a vector $\boldsymbol{u}$ such that $M_j^\top \boldsymbol{u} + \boldsymbol{b}_j' = 0,\ j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$. Combining the above two arguments, we have

$$M_j(M_j^\top \boldsymbol{u} + \boldsymbol{b}_j) = M_j(-\boldsymbol{b}_j' + \boldsymbol{b}_j) = 0 \quad \text{for all} \quad j = 1, \ldots, \left\lfloor \frac{d}{r} \right\rfloor.$$

It completes the proof. $\square$

According to the above corollary, if $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ is generalized affine phase retrievable in $\mathbb{R}^d$ then $m \ge d + \lfloor \frac{d}{r} \rfloor$. We next show the bound $d + \lfloor \frac{d}{r} \rfloor$ is tight provided $r \mid d$. To this end, we introduce the following lemma:

**Lemma 3.3.** *Suppose that $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{r+1} \in \mathbb{R}^r$ satisfy*

$$\mathrm{span}\{\boldsymbol{b}_2 - \boldsymbol{b}_1, \boldsymbol{b}_3 - \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{r+1} - \boldsymbol{b}_1\} = \mathbb{R}^r. \tag{3.2}$$

*Then $\boldsymbol{x} = \boldsymbol{y}$ if and only if $\|\boldsymbol{x} + \boldsymbol{b}_j\|_2 = \|\boldsymbol{y} + \boldsymbol{b}_j\|_2$ for all $j = 1, \ldots, r+1$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^r$.*

**Proof.** We denote $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y} \in \mathbb{R}^r$ and $t := (\|\boldsymbol{x}\|_2^2 - \|\boldsymbol{y}\|_2^2)/2$. Then $\|\boldsymbol{x} + \boldsymbol{b}_j\|_2 = \|\boldsymbol{y} + \boldsymbol{b}_j\|_2$ is equivalent to $\boldsymbol{b}_j^\top \boldsymbol{z} + t = 0$ for all $j = 1, \ldots, r+1$. To this end, we just need to show that $\|\boldsymbol{x} + \boldsymbol{b}_j\|_2 = \|\boldsymbol{y} + \boldsymbol{b}_j\|_2$ for all $j = 1, \ldots, r+1$ implies $\boldsymbol{x} = \boldsymbol{y}$. According to (3.2), the linear system

$$\begin{pmatrix} \boldsymbol{b}_1^\top & 1 \\ \vdots & \vdots \\ \boldsymbol{b}_{r+1}^\top & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{z} \\ t \end{pmatrix} = 0$$

has only zero solution, i.e., $(\boldsymbol{z}, t) = 0$, which implies $\boldsymbol{x} = \boldsymbol{y}$. $\square$

**Theorem 3.4.** *Suppose that $r \in \mathbb{Z}_{\ge 1}$ and $m \ge d + \lfloor \frac{d}{r} \rfloor + \epsilon_{d,r}$ where $\epsilon_{d,r} = 0$ if $d/r \in \mathbb{Z}$ and $1$ if $d/r \notin \mathbb{Z}$. Then there exists $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{R}^{d \times r} \times \mathbb{R}^r$ which has generalized affine phase retrieval property in $\mathbb{R}^d$.*

**Proof.** We set

$$T_t := \{(t-1)r+1, \dots, tr\}, \quad t = 1, \dots, \left\lfloor \frac{d}{r} \right\rfloor$$

and

$$T_{\lfloor \frac{d}{r} \rfloor + 1} := \left\{ r \left\lfloor \frac{d}{r} \right\rfloor + 1, \dots, d \right\}.$$

Note that if $d/r$ is an integer, then $T_{\lfloor \frac{d}{r} \rfloor+1} = \emptyset$. For $\boldsymbol{x} \in \mathbb{R}^d$, set $\boldsymbol{x}_{T_t} := \boldsymbol{x}\mathbb{I}_{T_t}$ where $\mathbb{I}_{T_t}$ denotes the indicator function of the set $T_t$ (namely $\mathbb{I}_{T_t}(s) = 1$ if $s \in T_t$ and 0 if $s \notin T_t$). Similarly, we use $(M_j)_{T_t} \in \mathbb{R}^{r\times r}$ to denote a submatrix of $M_j \in \mathbb{R}^{d\times r}$ with row indexes in $T_t$. Let $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ be the set of measurements which satisfies the following conditions:

(i) The matrix $(M_j)_{T_t} = I_r$ and $(M_j)_{[d]\setminus T_t}$ is a zero matrix for $j = (t-1)(r+1) + 1, \dots, t(r+1)$ and $t = 1, \dots, \lfloor d/r \rfloor$, where $I_r \in \mathbb{R}^{r\times r}$ is the identity matrix.
(ii) Set $\boldsymbol{b}_{(t-1)(r+1)+k} = \boldsymbol{b}'_k$ for $k = 1, \dots, r+1, t = 1, \dots, \lfloor d/r \rfloor$. The vectors $\boldsymbol{b}'_1, \dots, \boldsymbol{b}'_{r+1} \in \mathbb{R}^r$ satisfy $\operatorname{span}\{\boldsymbol{b}'_2 - \boldsymbol{b}'_1, \boldsymbol{b}'_3 - \boldsymbol{b}'_1, \dots, \boldsymbol{b}'_{r+1} - \boldsymbol{b}'_1\} = \mathbb{R}^r$.

Then, based on Lemma 3.3, for each $t = 1, \dots, \lfloor d/r \rfloor$, we can recover $\boldsymbol{x}_{T_t}$ from $\|M_j^\top \boldsymbol{x} + \boldsymbol{b}_j\|_2, j = (t-1)(r+1)+1, \dots, t(r+1)$. Hence, when $d/r \in \mathbb{Z}$, we can recover $\boldsymbol{x} = \boldsymbol{x}_{T_1} + \cdots + \boldsymbol{x}_{T_{(d/r+1)}}$ from $\|M_j \boldsymbol{x} + \boldsymbol{b}_j\|_2, j = 1, \dots, m$ where $m = (r+1)\lfloor d/r \rfloor = d + \lfloor d/r \rfloor$.

When $d/r$ is not an integer, we need consider the recovery of $\boldsymbol{x}_{T_{\lfloor d/r \rfloor+1}}$. Note that $\#T_{\lfloor d/r \rfloor+1} = d - r\lfloor d/r \rfloor$. Similar as before, we can construct matrix $M_j \in \mathbb{R}^{d\times r}$ and $\boldsymbol{b}_j \in \mathbb{R}^r, j = \lfloor d/r \rfloor (r+1) + 1, \dots, \lfloor d/r \rfloor + d + 1$ so that one can recover $\boldsymbol{x}_{T_{\lfloor d/r \rfloor+1}}$ from $\|M_j^\top \boldsymbol{x} + \boldsymbol{b}_j\|_2, j = \lfloor d/r \rfloor (r+1)+1, \dots, \lfloor d/r \rfloor + d+1$. Combining the measurement matrices above, we obtain the measurement number $m = \lfloor d/r \rfloor (r+1) + d - r\lfloor d/r \rfloor + 1 = d + \lfloor d/r \rfloor + 1$ is sufficient to recover $\boldsymbol{x}$ provided $d/r$ is not an integer. □

**Remark 3.5.** If we take $r = d$ in Theorem 3.4, we can construct $m = d+1$ matrices $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ so that $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = (\|M_1^* \boldsymbol{x} + \boldsymbol{b}_1\|_2^2, \dots, \|M_m^* \boldsymbol{x} + \boldsymbol{b}_m\|_2^2)$ is injective on $\mathbb{R}^d$. Hence, generalized affine phase retrieval can achieve the lower bound $m = d+1$ which is presented in Theorem 2.1.

As shown in [20, Theorem 2.3], the set of measurement matrices which has generalize phase retrieval property is an open set. The following theorem shows that the set of $\mathcal{A}$ having generalized *affine* phase retrieval property is *not* an open set in $\mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$. The result shows a difference between generalized phase retrieval and generalized affine phase retrieval.

**Theorem 3.6.** *Let $r \in \mathbb{Z}_{\geq 1}$ and $m \geq d + \lfloor \frac{d}{r} \rfloor + \epsilon_{d,r}$ where $\epsilon_{d,r} = 0$ if $d/r \in \mathbb{Z}$ and 1 if $d/r \notin \mathbb{Z}$. Then the set of generalized affine phase retrieval $\{(M_1, \boldsymbol{b}_1), \ldots (M_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ is not an open set in $\mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$.*

**Proof.** To this end, we only need to find a measurement set $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ which has generalized affine phase retrieval property in $\mathbb{R}^d$, but for any $\epsilon > 0$ there exists a small perturbation measurement set $\{(\widetilde{M}_1, \boldsymbol{b}_1), \ldots, (\widetilde{M}_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ with $\|M_j - \widetilde{M}_j\|_F \leq \epsilon$ which is not generalized affine phase retrievable.

We first consider the case where $r = d$. Without loss of generality we only need to consider the case $m = d + 1$ (for the case where $m > d + 1$, we just take $(M_j, \boldsymbol{b}_j) = 0$ for $j = d + 2, \ldots, m$). Set $M_j := I_d, j = 1, \ldots, d + 1$, and assume that $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{d+1} \in \mathbb{R}^d$ satisfy

$$\mathrm{span}\{\boldsymbol{b}_2 - \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{d+1} - \boldsymbol{b}_1\} = \mathbb{R}^d.$$

Here, we also require that the first entries of $\boldsymbol{b}_2, \ldots, \boldsymbol{b}_{d+1} \in \mathbb{R}^r$ are zero, i.e., $b_{2,1} = \cdots = b_{d+1,1} = 0$. According to Lemma 3.3, the measurement set $\{(M_1, \boldsymbol{b}_1), \ldots, (M_{d+1}, \boldsymbol{b}_{d+1})\} \in \mathbb{R}^{(d+1)(d\times d)} \times \mathbb{R}^{(d+1)d}$ has generalized affine phase retrievable property in $\mathbb{R}^d$.

We perturb $M_1$ to $\widetilde{M}_1 = I_d + \delta b_{1,1} E_{21}$, where $E_{21}$ denotes the matrix with $(2, 1)$-th entry being 1 and all other entries being 0 and $\delta > 0$. Furthermore, we let $\widetilde{M}_j = M_j$ for $j = 2, \ldots, d + 1$. Then $\{(\widetilde{M}_1, \boldsymbol{b}_1), \ldots, (\widetilde{M}_{d+1}, \boldsymbol{b}_{d+1})\} \in \mathbb{R}^{(d+1)(d\times r)} \times \mathbb{R}^{(d+1)r}$ is not generalized affine phase retrievable. To see this, we let $\boldsymbol{x} = (b_{1,1}, -1/\delta, 0, \ldots, 0)^\top$ and $\boldsymbol{y} = (-b_{1,1}, -1/\delta, 0, \ldots, 0)^\top$. It is easy to check that

$$\|\widetilde{M}_j^\top \boldsymbol{x} + \boldsymbol{b}_j\|_2 = \|\widetilde{M}_j^\top \boldsymbol{y} + \boldsymbol{b}_j\|_2 \quad j = 1, \ldots, d + 1.$$

By taking $\delta$ sufficiently small, we will have $\|M_j - \widetilde{M}_j\|_F \leq \epsilon$, which completes the proof for the case where $r = d$.

We next consider the case where $r \leq d - 1$. Similar with the proof of Theorem 3.4, we set

$$T_t := \{(t - 1)r + 1, \ldots, tr\}, \quad t = 1, \ldots, \left\lfloor \frac{d}{r} \right\rfloor$$

and

$$T_{\lfloor \frac{d}{r} \rfloor + 1} := \left\{ r \left\lfloor \frac{d}{r} \right\rfloor + 1, \ldots, d \right\}.$$

For $m = d + \lfloor \frac{d}{r} \rfloor + \epsilon_{d,r}$, we require that $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\}$ satisfies the conditions (i) and (ii) in the proof of Theorem 3.4. We furthermore require that the first entries of $\boldsymbol{b}_2, \ldots, \boldsymbol{b}_m$ are 0, i.e., $b_{2,1} = \cdots = b_{m,1} = 0$. Note that $(M_1)_{T_1} = I_r$. We perturb $(M_1)_{T_1}$ to $(\widetilde{M}_1)_{T_1} = I_r + \delta b_{1,1} E_{21}$ and $\widetilde{M}_j = M_j, j = 2, \ldots, m$. Then similar as before, $(\widetilde{M}_j, \boldsymbol{b}_j)_{j=1}^m$

does not have affine phase retrieval property but we will have $\|M_j - \widetilde{M}_j\|_F \le \epsilon$ by taking $\delta$ sufficiently small. We complete the proof for $r \le d-1$. $\square$

The following theorem shows that if the measurement number $m \ge 2d$, then a generic $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ has generalized affine phase retrieval property in $\mathbb{R}^d$.

**Theorem 3.7.** *Let $m \ge 2d$ and $r \in \mathbb{Z}_{\ge 1}$. Then a generic $\mathcal{A} = \{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ has generalized affine phase retrieval property in $\mathbb{R}^d$.*

To prove this theorem, we introduce some notations and lemmas. First, recall that

$$y_j = \|M_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2^2 = \tilde{\boldsymbol{x}}^* A_j \tilde{\boldsymbol{x}},\ j = 1, \ldots, m,$$

where

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \qquad \text{and} \qquad A_j = \begin{pmatrix} M_j M_j^* & M_j \boldsymbol{b}_j \\ (M_j \boldsymbol{b}_j)^* & \boldsymbol{b}_j^* \boldsymbol{b}_j \end{pmatrix}. \tag{3.3}$$

Thus, the map $\mathbf{M}_{\mathcal{A}}$ can be rewritten as

$$\begin{aligned} \mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) &:= (\|M_1^* \boldsymbol{x} + \boldsymbol{b}_1\|_2^2, \ldots, \|M_m^* \boldsymbol{x} + \boldsymbol{b}_m\|_2^2) \\ &= (\mathrm{tr}(A_1 \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^*), \ldots, \mathrm{tr}(A_m \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^*)). \end{aligned}$$

For $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$, we define the map $\mathbf{T} : \mathbb{C}^{(d+1)\times(d+1)} \to \mathbb{C}^m$ by

$$\mathbf{T}(Q) := (\mathrm{tr}(A_1^* Q), \ldots, \mathrm{tr}(A_m^* Q)). \tag{3.4}$$

**Lemma 3.8.** *Suppose that $r \in \mathbb{Z}_{\ge 1}$. Then $\mathcal{A} = \{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$ is not generalized affine phase retrievable if and only if there exists nonzero $Q \in \mathbb{R}^{(d+1)\times(d+1)}$ satisfies*

$$\begin{aligned} &Q^\top = Q, \quad Q_{d+1,d+1} = 0, \quad \mathrm{rank}(Q) \le 2, \\ &\mathbf{T}(Q) = 0, \quad Q_{1,d+1}^2 + \cdots + Q_{d,d+1}^2 = 1. \end{aligned} \tag{3.5}$$

**Proof.** Assume that $\mathcal{A}$ is not generalized affine phase retrievable, and then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ with $\boldsymbol{x} \ne \boldsymbol{y}$ such that $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = \mathbf{M}_{\mathcal{A}}(\boldsymbol{y})$. It implies that

$$\mathbf{T}(\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top - \tilde{\boldsymbol{y}}\tilde{\boldsymbol{y}}^\top) = 0,$$

where

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}, \qquad \tilde{\boldsymbol{y}} = \begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix}.$$

Take $Q := \lambda(\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top - \tilde{\boldsymbol{y}}\tilde{\boldsymbol{y}}^\top)$ where $\lambda = 1/\|\boldsymbol{x} - \boldsymbol{y}\|_2^2 \in \mathbb{R}$ is a constant. Then $Q$ is a nonzero matrix which satisfies (3.5).

We next assume there exists a nonzero $Q_0$ satisfies (3.5). According to the spectral decomposition theorem, we have

$$Q_0 = \lambda_1 \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^\top - \lambda_2 \tilde{\boldsymbol{v}}\tilde{\boldsymbol{v}}^\top$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}$ are normalized orthogonal vectors in $\mathbb{R}^{d+1}$. Since $(Q_0)_{d+1,d+1} = 0$, which gives that

$$\lambda_1 \tilde{u}_{d+1}^2 - \lambda_2 \tilde{v}_{d+1}^2 = 0.$$

Thus $\lambda_1$ and $\lambda_2$ have the same sign. We claim that $\lambda_1\lambda_2 \neq 0$ and $\tilde{u}_{d+1}\tilde{v}_{d+1} \neq 0$. Indeed, if $\lambda_2 = 0$, then $\tilde{u}_{d+1} = 0$. Hence, we obtain $(Q_0)_{1,d+1} = \cdots = (Q_0)_{d+1,d+1} = 0$ which contradicts with (3.5). So, $\lambda_2 \neq 0$. Similarly, we can show $\lambda_1 \neq 0$, $\tilde{u}_{d+1} \neq 0$ and $\tilde{v}_{d+1} \neq 0$. We take $\tilde{\boldsymbol{x}} := \tilde{\boldsymbol{u}}/\tilde{u}_{d+1}$ and $\tilde{\boldsymbol{y}} := \tilde{\boldsymbol{v}}/\tilde{v}_{d+1}$, and then $Q_0$ can be rewritten as

$$Q_0 = \lambda_1 \tilde{u}_{d+1}^2 \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top - \lambda_2 \tilde{v}_{d+1}^2 \tilde{\boldsymbol{y}}\tilde{\boldsymbol{y}}^\top = c(\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top - \tilde{\boldsymbol{y}}\tilde{\boldsymbol{y}}^\top)$$

where $c = \lambda_1 \tilde{u}_{d+1}^2 = \lambda_2 \tilde{v}_{d+1}^2 \in \mathbb{R}$ is a constant. Since $\mathbf{T}(Q_0) = 0$, it gives that $\mathbf{T}(\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^\top) = \mathbf{T}(\tilde{\boldsymbol{y}}\tilde{\boldsymbol{y}}^\top)$. We write $\tilde{\boldsymbol{x}} = (\boldsymbol{x}, 1)^\top$ and $\tilde{\boldsymbol{y}} = (\boldsymbol{y}, 1)^\top$ and then $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = \mathbf{M}_{\mathcal{A}}(\boldsymbol{y})$ which implies that $\mathcal{A}$ is not generalized affine phase retrievable. □

**Proof of Theorem 3.7 .** We use $\mathcal{G}_{m,d,r}$ to denote the subset of

$$(M_1, \boldsymbol{b}_1, \ldots, M_m, \boldsymbol{b}_m, Q) \in \mathbb{C}^{d\times r} \times \mathbb{C}^r \times \cdots \times \mathbb{C}^{d\times r} \times \mathbb{C}^r \times \mathbb{C}^{(d+1)\times(d+1)},$$

which satisfies the following property:

$$\begin{aligned} &Q^\top = Q, \quad Q_{d+1,d+1} = 0, \quad \operatorname{rank}(Q) \leq 2, \\ &\mathbf{T}(Q) = 0, \quad Q_{1,d+1}^2 + \cdots + Q_{d+1,d+1}^2 = 1. \end{aligned}$$

The $\mathcal{G}_{m,d,r}$ is a well defined complex affine variety because the defining equations are polynomials in each set of variables. We next consider the dimension of this complex affine variety $\mathcal{G}_{m,d,r}$. To this end, let $\pi_1$ be projections on the first $2m$ coordinates of $\mathcal{G}_{m,d,r}$, i.e.,

$$\pi_1(M_1, \boldsymbol{b}_1, \ldots, M_m, \boldsymbol{b}_m, Q) = (M_1, \boldsymbol{b}_1, \ldots, M_m, \boldsymbol{b}_m).$$

Similarly, we can define $\pi_2$ by

$$\pi_2(M_1, \boldsymbol{b}_1, \ldots, M_m, \boldsymbol{b}_m, Q) = Q.$$

We claim that $\pi_2(\mathcal{G}_{m,d,r}) = \mathcal{L}_d$ where

$$\mathcal{L}_d := \{Q \in \mathbb{C}^{(d+1)\times(d+1)} : Q^\top = Q,\ Q_{d+1,d+1} = 0,\ \operatorname{rank}(Q) \le 2, \\ Q_{1,d+1}^2 + \cdots + Q_{d+1,d+1}^2 = 1\}.$$

Indeed, for any fixed $Q' \in \mathcal{L}_d$, there exist $\{(M_j', \boldsymbol{b}_j')\}_{j=1}^m \in \mathbb{C}^{d\times r} \times \mathbb{C}^r$ satisfying $\mathbf{T}(Q') = 0$, because for each $j$ the equation $\operatorname{tr}((A_j')^* Q') = 0$ is a polynomial for the variables $(M_j', \boldsymbol{b}_j')$. Here, each matrix $A_j'$ is defined by $(M_j', \boldsymbol{b}_j')$ as (3.3). This implies that $(M_1', \boldsymbol{b}_1', \ldots, M_m', \boldsymbol{b}_m', Q') \in \mathcal{G}_{m,d,r}$ and $\pi_2(M_1', \boldsymbol{b}_1', \ldots, M_m', \boldsymbol{b}_m', Q') = Q'$. Thus we have $\pi_2(\mathcal{G}_{m,d,r}) = \mathcal{L}_d$. Note that $\mathcal{L}_d \subset \mathbb{C}^{(d+1)\times(d+1)}$ is an affine variety with dimension $2d-1$ and hence $\dim(\pi_2(\mathcal{G}_{m,d,r})) = 2d-1$.

We next consider the dimension of the preimage $\pi_2^{-1}(Q_0) \in \mathbb{C}^{d\times r} \times \mathbb{C}^r \times \cdots \times \mathbb{C}^{d\times r} \times \mathbb{C}^r$ for a fixed nonzero $Q_0 \in \mathcal{L}_d$. For each pair $(M_j, \boldsymbol{b}_j) \in \mathbb{C}^{d\times r} \times \mathbb{C}^r$, the equation $\operatorname{tr}(M_j^* Q_0) = 0$ defines a hypersurface of dimension $dr + r - 1$ in $\mathbb{C}^{d\times r} \times \mathbb{C}^r$. Hence, the preimage $\pi_2^{-1}(Q_0)$ has dimension $m(dr + r - 1)$. Then, according to [12, Cor.11.13]

$$\begin{aligned}\dim(\mathcal{G}_{m,d,r}) &= \dim(\pi_2(\mathcal{G}_{m,d,r})) + \dim(\pi_2^{-1}(Q_0)) \\ &= m(dr + r - 1) + 2d - 1.\end{aligned}$$

If $m \ge 2d$, then

$$\dim(\pi_1(\mathcal{G}_{m,d,r})) \le \dim(\mathcal{G}_{m,d,r}) = m(dr + r - 1) + 2d - 1 < m(dr + r).$$

Hence,

$$\dim_{\mathbb{R}}((\pi_1(\mathcal{G}_{m,d,r}))_{\mathbb{R}}) \le \dim(\pi_1(\mathcal{G}_{m,d,r})) < m(dr + r) = \dim(\mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}),$$

which implies that $(\pi_1(\mathcal{G}_{m,d,r}))_{\mathbb{R}}$ lies in a sub-manifold of $\mathbb{R}^{m(d\times r)} \times \mathbb{R}^{mr}$. Here, the first inequality follows from [7]. However, Lemma 3.8 implies that $(\pi_1(\mathcal{G}_{m,d,r}))_{\mathbb{R}}$ contains precisely these $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\}$ which is not generalized affine phase retrieval in $\mathbb{R}^d$. Hence, we arrive at conclusion. □

## 4. Generalized affine phase retrieval for complex signals

We consider the complex case in this section. Then for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^d$, we have

$$\|M^*\boldsymbol{x} + \boldsymbol{b}\|_2^2 - \|M^*\boldsymbol{y} + \boldsymbol{b}\|_2^2 = 4\Re\left(\boldsymbol{u}^* M M^* \boldsymbol{v} + (M\boldsymbol{b})^* \boldsymbol{v}\right) \tag{4.1}$$

where $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y})$ and $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{y})$. Here, we use $\Re(c)$ to denote the real part of a complex number $c$.

**Theorem 4.1.** *Suppose that* $r \in \mathbb{Z}_{\geq 1}$. *Let* $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{C}^{d\times r} \times \mathbb{C}^r$. *Then the followings are equivalent:*

(1) *$\mathcal{A}$ has the generalize affine phase retrieval in $\mathbb{C}^d$.*
(2) *For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^d$ and $\boldsymbol{v} \neq 0$, there exists a $j$ with $1 \leq j \leq m$ such that*

$$\Re(\boldsymbol{u}^* M_j M_j^* \boldsymbol{v} + (M_j \boldsymbol{b}_j)^* \boldsymbol{v}) \neq 0.$$

(3) *Viewing $\mathbf{M}_{\mathcal{A}}$ as a map $\mathbb{R}^{2d} \to \mathbb{R}^m$, the real Jacobian of $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x})$ has rank $2d$ for all $\boldsymbol{x} \in \mathbb{R}^{2d}$.*

**Proof.** (1)⇔(2). We first show that (2) ⇒ (1). We assume that (1) does not hold. Then there exist $\boldsymbol{x} \neq \boldsymbol{y}$ in $\mathbb{C}^d$ such that $\mathbf{M}_{\mathcal{A}}(\boldsymbol{x}) = \mathbf{M}_{\mathcal{A}}(\boldsymbol{y})$. From (4.1) for all $j$ we have

$$\|M_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2^2 - \|M_j^* \boldsymbol{y} + \boldsymbol{b}_j\|_2^2 = 4\Re(\boldsymbol{u}^* M_j M_j^* \boldsymbol{v} + (M_j \boldsymbol{b})^* \boldsymbol{v}) = 0.$$

Noting that $\boldsymbol{v} \neq 0$, we conclude a contradiction with (2), which implies (1) holds. The converse also follows from the similar argument.

(2)⇔(3). Note that $M_j M_j^*$ is a Hermitian matrix and we can write $M_j M_j^* = B_j + iC_j$ with $B_j, C_j \in \mathbb{R}^{d\times d}$ and $B_j^\top = B_j, C_j^\top = -C_j$. Let

$$F_j = \begin{pmatrix} B_j & -C_j \\ C_j & B_j \end{pmatrix}.$$

Then for any $\boldsymbol{u} = \boldsymbol{u}_R + i\boldsymbol{u}_I \in \mathbb{C}^d$, we have

$$\|M_j^* \boldsymbol{u} + \boldsymbol{b}_j\|_2^2 = \tilde{\boldsymbol{u}}^\top F_j \tilde{\boldsymbol{u}} + 2\tilde{\mathbf{c}}_j^\top \tilde{\boldsymbol{u}} + \boldsymbol{b}_j^* \boldsymbol{b}_j,$$

where

$$\tilde{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u}_R \\ \boldsymbol{u}_I \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{c}}_j = \begin{bmatrix} (M_j \boldsymbol{b}_j)_R \\ (M_j \boldsymbol{b}_j)_I \end{bmatrix}.$$

The real Jacobian $J(\boldsymbol{u})$ of the map $\mathbf{M}_{\mathcal{A}}$ at $\boldsymbol{u} \in \mathbb{C}^d$ is exactly

$$J(\boldsymbol{u}) = 2[F_1 \tilde{\boldsymbol{u}} + \tilde{\mathbf{c}}_1, \ldots, F_m \tilde{\boldsymbol{u}} + \tilde{\mathbf{c}}_m].$$

For any $\boldsymbol{v} = \boldsymbol{v}_R + i\boldsymbol{v}_I \in \mathbb{C}^d$, we have

$$2\Re(\boldsymbol{u}^* M_j M_j^* \boldsymbol{v} + (M_j \boldsymbol{b}_j)^* \boldsymbol{v}) = [\boldsymbol{v}_R^\top, \boldsymbol{v}_I^\top] J_j(\boldsymbol{u}), \tag{4.2}$$

where $J_j(\boldsymbol{u})$ denotes the $j$-column of $J(\boldsymbol{u})$, $\boldsymbol{v}_R$ and $\boldsymbol{v}_I$ denote the real and imaginary part of $\boldsymbol{v}$, respectively. Thus it is clear that (2) and (3) are equivalent. □

**Corollary 4.2.** *Let $r \in \mathbb{Z}_{\geq 1}$ and $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{C}^{d\times r} \times \mathbb{C}^r$. If $\mathcal{A}$ has generalized affine phase retrievable property in $\mathbb{C}^d$ then $m \geq 2d + \lfloor d/r \rfloor$.*

**Proof.** To this end, we just need to show that $\mathcal{A}$ does not have generalized affine phase retrievable property in $\mathbb{C}^d$ provided $m \leq 2d + \lfloor d/r \rfloor - 1$. We first claim that there exists a $\mathbf{u}_0 \in \mathbb{C}^d$ such that $M_j^*\mathbf{u}_0 + \boldsymbol{b}_j = 0$ for all $j = 1, \ldots, \lfloor d/r \rfloor$. Fix $\mathbf{u}_0$, the following system are homogeneous linear equations for the variables $\boldsymbol{v}_R, \boldsymbol{v}_I \in \mathbb{R}^d$:

$$\Re((M_j M_j^* \mathbf{u}_0 + M_j \boldsymbol{b}_j)^* \boldsymbol{v}) = 0, \quad j = \lfloor d/r \rfloor + 1, \ldots, m. \tag{4.3}$$

Note that those equations have $2d$ real variables $\boldsymbol{v}_R, \boldsymbol{v}_I$, but the number of equations is at most $2d - 1$. It means that (4.3) must have a nontrivial solution $\boldsymbol{v}_0 \neq 0$. Hence, if $m \leq 2d + \lfloor d/r \rfloor - 1$, then there exist $\mathbf{u}_0, \boldsymbol{v}_0 \in \mathbb{C}^d$ with $\boldsymbol{v}_0 \neq 0$ so that

$$\Re(\boldsymbol{u}_0^* M_j M_j^* \boldsymbol{v}_0 + (M_j \boldsymbol{b}_j)^* \boldsymbol{v}_0) = 0 \quad \text{for all } j = 1, \ldots, m$$

which contradicts with (2) in Theorem 4.1.

Finally, we need to prove the claim. For any $j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$, let $\boldsymbol{b}_j' \in \mathbb{C}^r$ be the projection vector of $\boldsymbol{b}_j$ onto the space spanned by the rows of $M_j$. Then we have $M_j(\boldsymbol{b}_j - \boldsymbol{b}_j') = 0$ for all $j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$. On the other hand, since each $\boldsymbol{b}_j'$ is in the space spanned by the columns of $M_j^*$, it means that there exists a vector $\boldsymbol{u}$ such that $M_j^* \boldsymbol{u} + \boldsymbol{b}_j' = 0,\ j = 1, \ldots, \lfloor \frac{d}{r} \rfloor$. Combining the above two arguments, we have

$$M_j(M_j^* \boldsymbol{u} + \boldsymbol{b}_j) = M_j(-\boldsymbol{b}_j' + \boldsymbol{b}_j) = 0 \quad \text{for all} \quad j = 1, \ldots, \left\lfloor \frac{d}{r} \right\rfloor.$$

It completes the proof. □

**Lemma 4.3.** *Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{C}^r$ and suppose that $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2r+1} \in \mathbb{C}^r$ satisfy*

$$\operatorname{span}_{\mathbb{R}}\{\boldsymbol{b}_2 - \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2r+1} - \boldsymbol{b}_1\} = \mathbb{C}^r. \tag{4.4}$$

*Then $\boldsymbol{z}_1 = \boldsymbol{z}_2$ if $\|\boldsymbol{z}_1 + \boldsymbol{b}_j\|_2 = \|\boldsymbol{z}_2 + \boldsymbol{b}_j\|_2$ for all $j = 1, \ldots, 2r+1$.*

**Proof.** We set $\boldsymbol{z}_R := \boldsymbol{z}_{1,R} - \boldsymbol{z}_{2,R} \in \mathbb{R}^r$, $\boldsymbol{z}_I := \boldsymbol{z}_{1,I} - \boldsymbol{z}_{2,I} \in \mathbb{R}^r$ and $t := (\|\boldsymbol{z}_1\|_2^2 - \|\boldsymbol{z}_2\|_2^2)/2$. Then $\|\boldsymbol{z}_1 + \boldsymbol{b}_j\|_2 = \|\boldsymbol{z}_2 + \boldsymbol{b}_j\|_2$ implies that $\boldsymbol{b}_{j,R}^\top \boldsymbol{z}_R + \boldsymbol{b}_{j,I}^\top \boldsymbol{z}_I + t = 0$ for all $j = 1, \ldots, 2r+1$. The (4.4) implies that the rank of the matrix

$$A = \begin{bmatrix} \boldsymbol{b}_{1,R}^\top & \boldsymbol{b}_{1,I}^\top & 1 \\ \vdots & \vdots & \vdots \\ \boldsymbol{b}_{2r+1,R}^\top & \boldsymbol{b}_{2r+1,I}^\top & 1 \end{bmatrix}$$

is $2r + 1$. And hence $A[\boldsymbol{z}_R^\top, \boldsymbol{z}_I^\top, t]^\top = 0$ has only zero solution which means that $\boldsymbol{z}_1 = \boldsymbol{z}_2$. □

Next, we will show that the bound $m \geq 2d + \lfloor d/r \rfloor$ is tight provided $r \mid d$.

**Theorem 4.4.** *Suppose that $m \geq 2d + \lfloor d/r \rfloor + \epsilon_{d,r}$ where $\epsilon_{d,r} = 0$ if $d/r \in \mathbb{Z}$ and 1 if $d/r \notin \mathbb{Z}$. There exists $\mathcal{A} = \{(M_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{C}^{d\times r} \times \mathbb{C}^r$ which has generalized affine phase retrieval property in $\mathbb{C}^d$.*

**Proof.** We set

$$T_t := \{(t-1)r + 1, \ldots, tr\}, \quad t = 1, \ldots, \left\lfloor \frac{d}{r} \right\rfloor$$

and

$$T_{\lfloor \frac{d}{r} \rfloor + 1} := \left\{ r \left\lfloor \frac{d}{r} \right\rfloor + 1, \ldots, d \right\}.$$

We first consider the case where $d/r$ is an integer with $T_{\lfloor \frac{d}{r} \rfloor + 1} = \emptyset$. Similarly to the real case, for $\boldsymbol{x} \in \mathbb{C}^d$, set $\boldsymbol{x}_{T_t} := \boldsymbol{x}\mathbb{I}_{T_t}$ where $\mathbb{I}_{T_t}$ denotes the indicator function of the set $T_t$. Let $(M_j)_{T_t} \in \mathbb{C}^{r\times r}$ denote a submatrix of $M_j \in \mathbb{C}^{d\times r}$ with row indexes in $T_t$. Let $(M_j, \boldsymbol{b}_j), j = 1, \ldots, m$, be the set of measurements which satisfies the following conditions:

(i) The matrix $(M_j)_{T_t} = I_r$ and $(M_j)_{[d]\setminus T_t}$ is a zero matrix for $j = (t-1)(2r+1) + 1, \ldots, t(2r+1)$ and $t = 1, \ldots, \lfloor d/r \rfloor$, where $I_r$ is $r \times r$ the identity matrix.
(ii) Set $\boldsymbol{b}_{(t-1)(2r+1)+k} = \boldsymbol{b}'_k$ for $k = 1, \ldots, 2r+1, t = 1, \ldots, \lfloor d/r \rfloor$. The vectors $\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_{2r+1} \in \mathbb{C}^r$ satisfy $\mathrm{span}_{\mathbb{R}}\{\boldsymbol{b}'_2 - \boldsymbol{b}'_1, \boldsymbol{b}'_3 - \boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_{2r+1} - \boldsymbol{b}'_1\} = \mathbb{C}^r$.

Then based on Lemma 4.3, for each $t = 1, \ldots, \lfloor d/r \rfloor$, we can recover $\boldsymbol{x}_{T_t}$ from $\|M_j^*\boldsymbol{x} + \boldsymbol{b}_j\|_2, j = (t-1)(2r+1)+1, \ldots, t(2r+1)$. Hence, when $d/r \in \mathbb{Z}$, we can recover $\boldsymbol{x} = \boldsymbol{x}_{T_1} + \cdots + \boldsymbol{x}_{T_{(d/r+1)}}$ from $\|M_j^*\boldsymbol{x} + \boldsymbol{b}_j\|_2, j = 1, \ldots, m$ where $m = (2r+1)\lfloor d/r \rfloor = 2d + \lfloor d/r \rfloor$.

When $d/r$ is not an integer, we need to consider the recovery of $\boldsymbol{x}_{T_{\lfloor d/r \rfloor+1}}$. Note that $\#T_{\lfloor d/r \rfloor+1} = d - r\lfloor d/r \rfloor$. Similar as before, we can construct matrix $M_j \in \mathbb{C}^{d\times r}$, and $\boldsymbol{b}_j \in \mathbb{C}^r, j = \lfloor d/r \rfloor (2r+1)+1, \ldots, \lfloor d/r \rfloor (2r+1)+2d-2r\lfloor d/r \rfloor+1$ so that one can recover $\boldsymbol{x}_{T_{\lfloor d/r \rfloor+1}}$ from $\|M_j^*\boldsymbol{x} + \boldsymbol{b}_j\|_2, j = \lfloor d/r \rfloor (2r+1)+1, \ldots, \lfloor d/r \rfloor (2r+1)+2d-2r\lfloor d/r \rfloor+1$. Combining the measurement matrices above, we obtain the measurement number $m = \lfloor d/r \rfloor (2r+1) + 2d - 2r\lfloor d/r \rfloor + 1 = 2d + \lfloor d/r \rfloor + 1$ is sufficient to recover $\boldsymbol{x}$ provided $d/r$ is not an integer. □

Similar to the real case, the set of $\mathcal{A} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ which can do generalized affine phase retrieval is not an open set.

**Theorem 4.5.** *Let $r \in \mathbb{Z}_{\geq 1}$ and $m \geq 2d + \lfloor \frac{d}{r} \rfloor + \epsilon_{d,r}$ where $\epsilon_{d,r} = 0$ if $d/r \in \mathbb{Z}$ and 1 if $d/r \notin \mathbb{Z}$. Then the set of generalized affine phase retrieval $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ is not an open set.*

**Proof.** We only need to find a measurement set $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ which has generalized affine phase retrieval property in $\mathbb{C}^d$, but for any $\epsilon > 0$ there exists a small perturbation measurement set $\{(\widetilde{M}_1, \boldsymbol{b}_1), \ldots, (\widetilde{M}_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ with $\|M_j - \widetilde{M}_j\|_F \leq \epsilon$ which is not generalized affine phase retrievable.

We first consider the case where $r = d$. Without loss of generality we only need to consider the case $m = 2d+1$ (for the case where $m > 2d+1$, we just take $(M_j, \boldsymbol{b}_j) = (\mathbf{0}, \mathbf{0})$ for $j = 2d+2, \ldots, m$). Set $M_j := I_d, j = 1, \ldots, 2d+1$, and

$$\boldsymbol{b}_j = \begin{cases} i\boldsymbol{e}_j & j = 1, \ldots, d \\ \boldsymbol{e}_j & j = d+1, \ldots, 2d \\ 0 & j = 2d+1 \end{cases}, \tag{4.5}$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is the canonical basis vectors in $\mathbb{C}^d$, i.e. the $j$th entry of $\boldsymbol{e}_j$ is 1 and other entries are 0. A simple observation is that $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2d+1} \in \mathbb{C}^d$ satisfy

$$\mathrm{span}_{\mathbb{R}}\{\boldsymbol{b}_2 - \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2d+1} - \boldsymbol{b}_1\} = \mathbb{C}^d.$$

According to Lemma 3.3, the measurement set $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^{2d+1}$ has generalized affine phase retrievable property in $\mathbb{C}^d$.

We perturb $M_1$ to $\widetilde{M}_1 = I_d + i\delta E_{12} - i\delta E_{21}$, where $E_{12}$ denotes the matrix with $(1,2)$-th entry being 1 and all other entries being 0 and $\delta > 0$. Furthermore, we let $\widetilde{M}_j = M_j$ for $j = 2, \ldots, 2d+1$. Then $\{(\widetilde{M}_j, \boldsymbol{b}_j)\}_{j=1}^m \subset \mathbb{C}^{d\times r} \times \mathbb{C}^r$ is not generalized affine phase retrievable. To see this, we let $\boldsymbol{x} = (i, -\frac{1}{2\delta}, 0, \ldots, 0)^\top$ and $\boldsymbol{y} = (-i, -\frac{1}{2\delta}, 0, \ldots, 0)^\top$. It is easy to check that

$$\|\widetilde{M}_j^* \boldsymbol{x} + \boldsymbol{b}_j\|_2 = \|\widetilde{M}_j^* \boldsymbol{y} + \boldsymbol{b}_j\|_2 \quad j = 1, \ldots, 2d+1.$$

By taking $\delta$ sufficiently small, we will have $\|M_j - \widetilde{M}_j\|_F \leq \epsilon$, which completes the proof for the case where $r = d$.

We next consider the case where $r \leq d-1$. Using the notations in Theorem 4.4, we set

$$T_t := \{(t-1)r+1, \ldots, tr\}, \quad t = 1, \ldots, \left\lfloor \frac{d}{r} \right\rfloor$$

and

$$T_{\lfloor \frac{d}{r} \rfloor + 1} := \left\{ r\left\lfloor \frac{d}{r} \right\rfloor + 1, \ldots, d \right\}.$$

For $m = 2d + \lfloor \frac{d}{r} \rfloor + \epsilon_{d,r}$, we require that $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ satisfies the conditions (i) and (ii) in the proof of Theorem 4.4, i.e.,

(i) The matrix $(M_j)_{T_t} = I_r$ and $(M_j)_{[d]\setminus T_t}$ is a zero matrix for $j = (t-1)(2r+1)+1, \ldots, t(2r+1)$ and $t = 1, \ldots, \lfloor d/r \rfloor$, where $I_r$ is the identity matrix with size $r \times r$.
(ii) Set $\boldsymbol{b}_{(t-1)(2r+1)+k} = \boldsymbol{b}'_k$ for $k = 1, \ldots, 2r+1, t = 1, \ldots, \lfloor d/r \rfloor$. The vectors $\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_{2r+1} \in \mathbb{C}^r$ satisfy $\mathrm{span}_{\mathbb{R}}\{\boldsymbol{b}'_2 - \boldsymbol{b}'_1, \boldsymbol{b}'_3 - \boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_{2r+1} - \boldsymbol{b}'_1\} = \mathbb{C}^r$.

Particularly, we require that $\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_{2r+1} \in \mathbb{C}^r$ are similarly defined by (4.5). Note that $(M_1)_{T_1} = I_r$. Similar as before, we perturb $(M_1)_{T_1}$ to $(\widetilde{M}_1)_{T_1} = I_r + i\delta E_{12} - i\delta E_{21}$ and $\widetilde{M}_j = M_j, j = 2, \ldots, m$. Then $\{(\widetilde{M}_j, \boldsymbol{b}_j)\}_{j=1}^m$ does not have affine phase retrieval property but we will have $\|M_j - \widetilde{M}_j\|_F \le \epsilon$ by taking $\delta$ sufficiently small, which completes the proof for $r \le d-1$. $\square$

**Theorem 4.6.** *Let $r \in \mathbb{Z}_{\ge 1}$ and $m \ge 4d-1$. Then a generic $\{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ has generalized affine phase retrieval property in $\mathbb{C}^d$.*

To this end, we introduce some lemmas.

**Lemma 4.7.** *Suppose that $r \in \mathbb{Z}_{\ge 1}$. Then $\mathcal{A} = \{(M_1, \boldsymbol{b}_1), \ldots, (M_m, \boldsymbol{b}_m)\} \in \mathbb{C}^{m(d\times r)} \times \mathbb{C}^{mr}$ is not generalized affine phase retrievable if and only if there exists nonzero $Q \in \mathbb{C}^{(d+1)\times(d+1)}$ satisfies*

$$\begin{aligned} &Q^* = Q, \quad Q_{d+1,d+1} = 0, \quad \mathrm{rank}(Q) \le 2, \quad \mathbf{T}(Q) = 0, \\ &Q_{1,d+1} \cdot Q_{d+1,1} + \cdots + Q_{d,d+1} \cdot Q_{d+1,d} = 1, \end{aligned} \tag{4.6}$$

*where the linear operator $\mathbf{T}$ is defined in (3.4).*

The proof of Lemma 4.7 is similar to the one of Lemma 3.8. We omit the detail here. To state conveniently, we use $\mathbb{C}^{d\times d}_{\mathrm{sym}}$ to denote the set of symmetric complex $d \times d$ matrices and use $\mathbb{C}^{d\times d}_{\mathrm{skew}}$ to denote the set of skew-symmetric complex $d \times d$ matrices.

**Definition 4.8.** Let $\mathcal{G}_{m,d,r}$ denote the set of $(U_1, \mathbf{c}_1, V_1, \mathbf{d}_1, \ldots, U_m, \mathbf{c}_m, V_m, \mathbf{d}_m, X, Y)$ where $U_j, V_j \in \mathbb{C}^{d\times r}, \mathbf{c}_j, \mathbf{d}_j \in \mathbb{C}^r$, $X \in \mathbb{C}^{(d+1)\times(d+1)}_{\mathrm{sym}}, Y \in \mathbb{C}^{(d+1)\times(d+1)}_{\mathrm{skew}}$ which satisfies the following properties:

$$\begin{aligned} &X_{d+1,d+1} = 0, \quad \mathrm{rank}(X + iY) \le 2, \quad \langle A_j, X + iY \rangle = 0, \quad j = 1, \ldots, m \\ &(X_{1,d+1} + iY_{1,d+1})(X_{d+1,1} + iY_{d+1,1}) + \cdots + (X_{d,d+1} + iY_{d,d+1})(X_{d+1,d} + iY_{d+1,d}) = 1, \end{aligned}$$

where

$$A_j = \begin{pmatrix} M_j M_j^* & M_j \boldsymbol{b}_j \\ (M_j \boldsymbol{b}_j)^* & \boldsymbol{b}_j^* \boldsymbol{b}_j \end{pmatrix}, \tag{4.7}$$

$M_j = U_j + iV_j$ and $\boldsymbol{b}_j = \mathbf{c}_j + i\mathbf{d}_j$.

Recall that $\mathrm{rank}(X + iY) \le 2$ is equivalent to the vanishing to all $3 \times 3$ minors of $X + iY$. Hence, we can view $\mathcal{G}_{m,d,r}$ as a complex affine variety. Next, we consider the dimension of $\mathcal{G}_{m,d,r}$.

**Lemma 4.9.** *The complex affine variety $\mathcal{G}_{m,d,r}$ has dimension $(2dr + 2r - 1)m + 4d - 2$.*

**Proof.** Let $\mathcal{G}'_{m,d,r}$ be the set of $(U_1, \mathbf{c}_1, V_1, \mathbf{d}_1, \dots, U_m, \mathbf{c}_m, V_m, \mathbf{d}_m, Q)$ where $U_j, V_j \in \mathbb{C}^{d\times r}, \mathbf{c}_j, \mathbf{d}_j \in \mathbb{C}^r$, $Q \in \mathbb{C}^{(d+1)\times(d+1)}$ which satisfies

$$\begin{aligned}&Q_{d+1,d+1} = 0, \quad \mathrm{rank}(Q) \le 2, \quad \langle A_j, Q\rangle = 0, j = 1, \dots, m\\ &Q_{1,d+1} \cdot Q_{d+1,1} + \cdots + Q_{d,d+1} \cdot Q_{d+1,d} = 1,\end{aligned}$$

where matrices $A_j$ are defined by (4.7). Note that $\mathcal{G}'_{m,d,r}$ is a well defined complex affine variety because the defining equations are polynomials in each set of variables. It is clear that $\mathcal{G}_{m,d,r}$ and $\mathcal{G}'_{m,d,r}$ are linear isomorphic since we can identify $\mathbb{C}^{d\times d}_{\mathrm{sym}} \times \mathbb{C}^{d\times d}_{\mathrm{skew}}$ with $\mathbb{C}^{d\times d}$ by the map $(X, Y) \mapsto X + iY = Q$. Indeed, any complex matrix $Q$ can be uniquely written as $Q = X + iY$ where $X = (Q + Q^\top)/2$ is a complex symmetric matrix and $Y = (Q - Q^\top)/2i$ is a complex skew-symmetric matrix. Hence, to this end, it is sufficient to consider the dimension of $\mathcal{G}'_{m,d,r}$.

We let $\pi_1$ and $\pi_2$ be projections on the first $4m$ coordinates and the last coordinate of $\mathcal{G}'_{m,d,r}$, respectively, i.e.,

$$\pi_1(U_1, \mathbf{c}_1, V_1, \mathbf{d}_1, \dots, U_m, \mathbf{c}_m, V_m, \mathbf{d}_m, Q) = (U_1, \mathbf{c}_1, V_1, \mathbf{d}_1, \dots, U_m, \mathbf{c}_m, V_m, \mathbf{d}_m)$$

and

$$\pi_2(U_1, \mathbf{c}_1, V_1, \mathbf{d}_1, \dots, U_m, \mathbf{c}_m, V_m, \mathbf{d}_m, Q) = Q.$$

We claim that $\pi_2(\mathcal{G}_{m,d,r}) = \mathcal{L}_d$ where

$$\begin{aligned}\mathcal{L}_d := \{&Q \in \mathbb{C}^{(d+1)\times(d+1)} : Q_{d+1,d+1} = 0,\\ &\mathrm{rank}(Q) \le 2,\ Q_{1,d+1} \cdot Q_{d+1,1} + \cdots + Q_{d,d+1} \cdot Q_{d+1,d} = 1\}.\end{aligned}$$

Indeed, for any fixed $Q' \in \mathcal{L}_d$, there exists $\{(U'_j, \mathbf{c}'_j, V'_j, \mathbf{d}'_j)\}_{j=1}^m \in \mathbb{C}^{d\times r} \times \mathbb{C}^r \times \mathbb{C}^{d\times r} \times \mathbb{C}^r$ satisfying $\langle A'_j, Q'\rangle = 0, j = 1, \dots, m$. Here, each matrix $A'_j$ is defined by $(U'_j, \mathbf{c}'_j, V'_j, \mathbf{d}'_j)$ as (4.7). It is because that for each $j$ the equation $\langle A'_j, Q'\rangle = 0$ is a polynomial which only contains variables $(U'_j, \mathbf{c}'_j, V'_j, \mathbf{d}'_j)$. Thus we have $\pi_2(\mathcal{G}'_{m,d,r}) = \mathcal{L}_d$. Note that $\mathcal{L}_d \subset \mathbb{C}^{(d+1)\times(d+1)}$ is an affine variety with dimension $4d - 2$ and hence $\dim(\pi_2(\mathcal{G}'_{m,d,r})) = 4d - 2$.

We next consider the dimension of the preimage $\pi_2^{-1}(Q_0)$ for a fixed nonzero $Q_0 \in \mathcal{L}_d$. For each pair $(U_j, \mathbf{c}_j, V_j, \mathbf{d}_j)$, the equation $\langle A_j, Q_0\rangle = 0$ defines a hypersurface of dimension $2dr + 2r - 1$ in $\mathbb{C}^{d\times r} \times \mathbb{C}^r \times \mathbb{C}^{d\times r} \times \mathbb{C}^r$. Hence, the preimage $\pi_2^{-1}(Q_0)$ has dimension $m(2dr + 2r - 1)$. Then, according to [12, Cor.11.13]

$$\begin{aligned}\dim(\mathcal{G}_{m,d,r}) = \dim(\mathcal{G}'_{m,d,r}) &= \dim(\pi_2(\mathcal{G}'_{m,d,r})) + \dim(\pi_2^{-1}(Q_0)) \\ &= m(2dr + 2r - 1) + 4d - 2. \quad \square\end{aligned}$$

**Proof of Theorem 4.6 .** For each $(M_j, \boldsymbol{b}_j) \in \mathbb{C}^{d\times r} \times \mathbb{C}^r$, we use $U_j, V_j$ and $\mathbf{c}_j, \mathbf{d}_j$ to denote the real and imaginary part of $M_j$ and $\boldsymbol{b}_j$, respectively. By Lemma 4.7, a tuple of real matrices $\{(U_j, \mathbf{c}_j, V_j, \mathbf{d}_j)\}_{j=1}^m$ for which the corresponding $\{(M_j, \boldsymbol{b}_j)\}_{j=1}^m$ does not have generalized affine phase retrieval property gives a point $\{(U_j, \mathbf{c}_j, V_j, \mathbf{d}_j)\}_{j=1}^m \in \pi_1((\mathcal{G}_{m,d,r})_{\mathbb{R}}) \subset (\pi_1(\mathcal{G}_{m,d,r}))_{\mathbb{R}}$. A simple observation is that, if $m \geq 4d - 1$, then

$$\dim(\pi_1(\mathcal{G}_{m,d,r})) \leq \dim(\mathcal{G}_{m,d,r}) = m(2dr + 2r - 1) + 4d - 2 < m(2dr + 2r).$$

Hence,

$$\dim_{\mathbb{R}}((\pi_1(\mathcal{G}_{m,d,r}))_{\mathbb{R}}) \leq \dim(\pi_1(\mathcal{G}_{m,d,r})) < m(2dr + 2r) = \dim(\mathbb{R}^{d\times r} \times \mathbb{R}^r \times \mathbb{R}^{d\times r} \times \mathbb{R}^r).$$

This implies that the set

$$\{(M_j, \boldsymbol{b}_j)_{j=1}^m \in \mathbb{C}^{d\times r} \times \mathbb{C}^r : (M_j, \boldsymbol{b}_j)_{j=1}^m \text{ does not have generalized affine phase retrieval property}\}$$

corresponds to a set $\{(U_j, \mathbf{c}_j, V_j, \mathbf{d}_j)\}_{j=1}^m$ which lies in a sub-manifold of $\mathbb{R}^{d\times r} \times \mathbb{R}^r \times \mathbb{R}^{d\times r} \times \mathbb{R}^r$. Hence, we arrive at conclusion. $\square$

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